Int. J. Solids Structures Vol. 20, No. 3, pp. 257-265, 1984
Printed in Great Britain.
0020-7683/84 \$3.00 + .00

NATURAL FREQUENCIES FOR OUT-OF-PLANE VIBRATIONS OF CONTINUOUS CURVED BEAMS CONSIDERING SHEAR AND ROTARY INERTIA

TUNG-MING WANG, ANDREA J. LASKEY† and MOHAMED F. AHMAD
Department of Civil Engineering, University of New Hampshire, Durham, NH 03824, U.S.A.

(*Received* 2 *September* 1982; *in revised form* 16 *June* 1983)

Abstract—The dynamic stiffness matrix for horizontally circular curved members of constant section is presented for determining natural frequencies of continuous curved beams undergoing out-of-plane vibrations. A three-span curved beam is provided to illustrate the application of the proposed method and to show the effects of transverse shear, rotary inertia and the opening angle of the arc on the natural frequencies of the beam.

1. INTRODUCTION

The dynamic response of curved beams vibrating out of their initial plane of curvature is of interest in many fields of engineering. Den Hartog[1] employed the Rayleigh–Ritz method to find the lowest natural frequency of circular arcs with clamped ends. His work was extended by Volterra and Morell[2, 3] and Wang[4] for vibrations of non-cirular arcs. Using the equations of motion derived by Love, Ojalvo[5] studied the coupled twist-bending vibrations of incomplete rings. Recently, Wang *et al.*[6] introducted the dynamic slope-deflection equations for finding the natural frequencies of continuous circular curved beams.

The elementary Bernoulli–Euler theory of flexural vibrations of beams has been known to be adequate for relatively slender beams at lower modes of vibation. For short and thin-webbed beams and for beams where higher modes are required, the Timoshenko theory[7] which considers the effects of shear deformation and rotary inertia provides a better approximation to the actual beam behavior.

There has been a great deal of work done on vibrations of curved beams according to the Timoshenko theory. The in-plane vibrations of circular rings with shear and rotary inertia effects being included were treated by Philipson[8]. Seidal and Erdelyi[9] studied the in-plane vibrations of non-thin rings subjected to the effects of bending, shear and extensional strain energies, together with translational and rotational kinetic energies. The effects of shear and rotary inertia on both in-plane and out-of-plane vibrations of rings were analyzed by Rao and Sundararajan[10, 11]. More recently, Wang and Guilbert[12] investigated the in-plane free vibrations of multispan curved beams considering rotary inertia and shear.

Of the studies just mentioned, only the effects of shear and rotary inertia on out-of-plane vibrations of simple curved beams have been considered. The purpose of this paper is to present a general method for the analysis of continuous curved beams including both rotary inertia and shear effects. In the present work, the dynamic stiffness matrix for a horizontally circular curved member in terms of rotations, angles of twist and vertical displacements has been derived. An example of a three-span curved beam undergoing out-of-plane vibrations is provided to illustrate the application of the proposed method and to demonstrate the effects of shear deformation, rotary inertia and the opening angle of the arc on the natural frequencies of the beam.

2. DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

Figure 1 shows the out-of-plane, small vibration of a horizontally circular curved element with the effects of damping and warping neglected. The expressions for the

†Now at Stone and Webster Engineering Corporation, Boston, MA 02177, U.S.A.

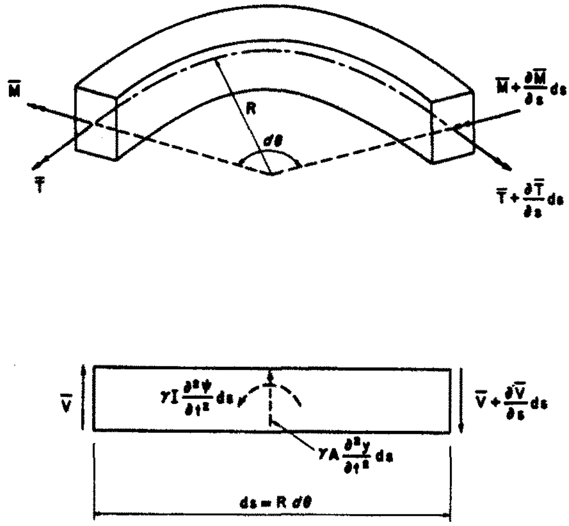

Fig. 1. Element of a horizontally curved member subjected to forces and moments.

bending moment, $\bar{M}$, and twisting moment, $\bar{T}$, of a curved beam can be expressed as [13]

$$\bar{M}(\theta, t) = \frac{EI}{R}\left(\phi - \frac{\partial\psi}{\partial\theta}\right) \tag{1}$$

$$\bar{T}(\theta, t) = \frac{C}{R}\left(\psi + \frac{\partial\phi}{\partial\theta}\right) \tag{2}$$

where EI is the flexural rigidity, C the torsional rigidity, ψ the bending slope, ϕ the angle of twist, R the radius of a circular member, θ the angular coordinate, and t the time.

The total angle between the deformed and undeformed center lines of the beam as shown in Fig. 2 is [7]

$$\frac{1}{R}\frac{\partial y}{\partial\theta} = \psi + \beta \tag{3}$$

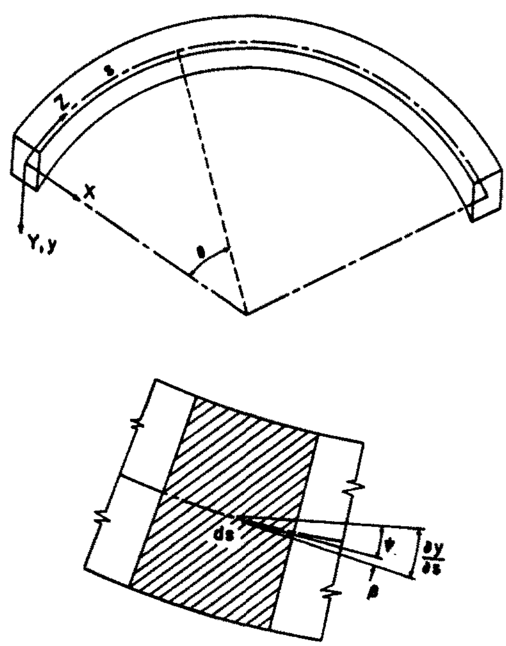

Fig. 2. Strain-displacement relation on a typical cross section.

where y is the vertical displacement and β the angular deformation due to shear. Thus, the transverse shear force $\bar{V}$ may be written as[14]

$$\bar{V}(\theta, t) = kAG\beta = kAG\left(\frac{1}{R}\frac{\partial y}{\partial \theta} - \psi\right) \tag{4}$$

where k is the cross-sectional shape factor, A the cross-sectional area, and G the modulus of rigidity.

The equilibrium conditions of the curved element shown in Fig. 1 give

$$\frac{\partial \bar{V}}{\partial \theta} - \gamma AR\frac{\partial^2 y}{\partial t^2} = 0 \tag{5}$$

$$\frac{\partial \bar{M}}{\partial \theta} - \bar{V}R + \bar{T} + \gamma IR\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{6}$$

$$\bar{M} - \frac{\partial \bar{T}}{\partial \theta} = 0 \tag{7}$$

where γ is the mass per unit volume and I the moment of inertia of cross section.

From eqns (1)–(7) one obtains

$$\begin{aligned}\frac{\partial^6 y}{\partial \theta^6} + 2\frac{\partial^4 y}{\partial \theta^4} + \frac{\partial^2 y}{\partial \theta^2} &= \left(\frac{\gamma R^2}{E} + \frac{\gamma R^2}{kG}\right)\frac{\partial^6 y}{\partial \theta^4 \partial t^2} - \left(\frac{\gamma^2 R^4}{EkG}\right)\frac{\partial^6 y}{\partial \theta^2 \partial t^4}\\ &+ \left(\frac{2\gamma R^2}{kG} - \frac{\gamma R^2}{\rho E} - \frac{\gamma AR^4}{EI}\right)\frac{\partial^4 y}{\partial \theta^2 \partial t^2} + \left(\frac{\gamma^2 R^4}{\rho EkG}\right)\frac{\partial^4 y}{\partial t^4} + \left(\frac{\gamma R^2}{kG} + \frac{\gamma AR^4}{\rho EI}\right)\frac{\partial^2 y}{\partial t^2}\end{aligned} \tag{8}$$

$$\begin{aligned}\phi R = \frac{\rho}{1+\rho}\Bigg\{&\frac{\partial^4 y}{\partial \theta^4} + \left(\frac{1+2\rho}{\rho}\right)\frac{\partial^2 y}{\partial \theta^2} - \left(\frac{\gamma R^2}{E} + \frac{\gamma R^2}{kG}\right)\frac{\partial^4 y}{\partial \theta^2 \partial t^2}\\ &+ \left(\frac{\gamma^2 R^4}{EkG}\right)\frac{\partial^4 y}{\partial t^4} + \left(\frac{\gamma AR^4}{EI} - \frac{1+2\rho}{\rho}\frac{\gamma R^2}{kG}\right)\frac{\partial^2 y}{\partial t^2}\Bigg\}\end{aligned} \tag{9}$$

$$\begin{aligned}&\left(1 + \frac{\rho EI}{kAGR^2}\right)\psi + \left(\frac{\gamma I}{kAG}\right)\frac{\partial^2 \psi}{\partial t^2}\\ &= \left(\frac{EI}{kAGR^3}\right)\frac{\partial^3 y}{\partial \theta^3} + \frac{1}{R}\frac{\partial y}{\partial \theta} - \left(\frac{\gamma EI}{k^2 AG^2 R}\right)\frac{\partial^3 y}{\partial \theta \partial t^2} - \frac{EI(1+\rho)}{kAGR^2}\frac{\partial \phi}{\partial \theta}\end{aligned} \tag{10}$$

where $\rho = C/(EI)$ is the stiffness parameter.

Assuming that the curved member is under the action of free vibration with a frequency p and letting

$$y(\theta, t) = Y(\theta)\,e^{ipt} \tag{11}$$

$$\phi(\theta, t) = \Phi(\theta)\,e^{ipt} \tag{12}$$

$$\psi(\theta, t) = \Psi(\theta)\,e^{ipt} \tag{13}$$

where $i = \sqrt{-1}$, and Y, Φ and Ψ are the normal functions for y, ϕ and ψ, respectively. Substituting the above equations into eqns (8)–(10) and omitting e^{ipt} yields

$$\begin{aligned}&Y^{VI} + (2 + b^2r^2 + b^2s^2)Y^{IV} + (1 - b^2 - b^2r^2/\rho + 2b^2s^2 + b^4r^2s^2)Y''\\ &+ (b^2/\rho + b^2s^2 - b^4r^2s^2/\rho)Y = 0\end{aligned} \tag{14}$$

$$\Phi R = \frac{\rho}{1+\rho}\left\{Y^{IV} + \left(\frac{1+2\rho}{\rho} + b^2r^2 + b^2s^2\right)Y'' + \left(\frac{1+2\rho}{\rho}b^2s^2 - b^2 + b^4r^2s^2\right)Y\right\} \quad (15)$$

$$\Psi R = \left(\frac{\rho s^2}{b^2r^2s^2 - \rho s^2 - 1}\right)\left\{Y^V + (2 + b^2r^2 + b^2s^2)Y''' + \left(2b^2s^2 - b^2 - \frac{1}{\rho s^2} + b^4r^2s^2\right)Y'\right\} \quad (16)$$

where b, r, s represent effects of bending, rotary inertia and shear deformation, respectively, and are given by

$$b^2 = \gamma AR^4p^2/(EI), \quad r^2 = I/(AR^2), \quad s^2 = EI/(kAGR^2) \quad (17)$$

and the primes for Y indicate differentiation with respect to θ.

The general solution of eqn (14) takes the form of

$$Y(\theta) = \sum_{n=1}^{6} c_n e^{\lambda_n \theta} \quad (18)$$

where c_n are constants to be determined from boundary conditions, and λ_n are the roots of the following auxillary equation

$$\lambda^6 + (2 + b^2r^2 + b^2s^2)\lambda^4 + (1 - b^2 - b^2r^2/\rho + 2b^2s^2 + b^4r^2s^2)\lambda^2$$
$$+ (b^2/\rho + b^2s^2 - b^4r^2s^2/\rho) = 0. \quad (19)$$

Substituting eqn (18) into eqns (15) and (16) yield

$$\Phi(\theta)\, R = \sum_{n=1}^{6} c_n w_n e^{\lambda_n \theta} \quad (20)$$

$$\Psi(\theta)\, R = \sum_{n=1}^{6} c_n z_n e^{\lambda_n \theta} \quad (21)$$

where

$$w_n = \frac{\rho}{1+\rho}\left\{\lambda_n^4 + \left(\frac{1+2\rho}{\rho} + b^2r^2 + b^2s^2\right)\lambda_n^2 + \left(\frac{1+2\rho}{\rho}b^2s^2 - b^2 + b^4r^2s^2\right)\right\} \quad (22)$$

$$z_n = \left(\frac{\rho s^2}{b^2r^2s^2 - \rho s^2 - 1}\right)\left\{\lambda_n^5 + (2 + b^2r^2 + b^2s^2)\lambda_n^3 + \left(2b^2s^2 - b^2 - \frac{1}{\rho s^2} + b^4r^2s^2\right)\lambda_n\right\}. \quad (23)$$

3. DYNAMIC STIFFNESS MATRIX FOR HORIZONTALLY CURVED MEMBER

Consider a horizontally circular curved member of constant cross-section subjected to harmonic displacements Ψ_a, Ψ_b, Φ_a, Φ_b, Y_a and Y_b as shown in Fig. 3.

Let

$$\bar{M}(\theta, t) = M(\theta)\, e^{ipt} \quad (24)$$

$$\bar{T}(\theta, t) = T(\theta)\, e^{ipt} \quad (25)$$

$$\bar{V}(\theta, t) = V(\theta)\, e^{ipt} \quad (26)$$

where M, T and V are the normal functions for $\bar{M}$, $\bar{T}$ and $\bar{V}$, respectively.

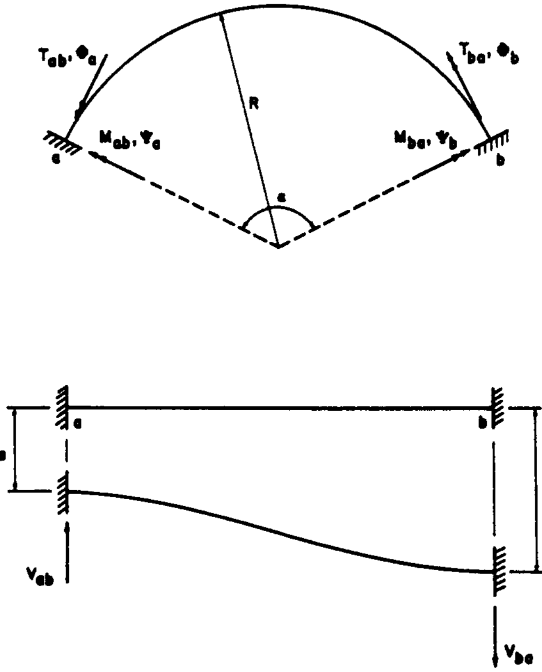

Fig. 3. Positive displacements, forces and moments with e^{ipt} omitted.

Introducing eqns (11)–(13) and (24)–(26) into eqns (1), (2) and (4) and omitting e^{ipt} give

$$M(\theta)=\frac{EI}{R}\{\Phi(\theta)-\Psi'(\theta)\} \tag{27}$$

$$T(\theta)=\frac{C}{R}\{\Psi(\theta)+\Phi'(\theta)\} \tag{28}$$

$$V(\theta)=\frac{kAG}{R}\{Y'(\theta)-R\Psi(\theta)\}. \tag{29}$$

Substituting eqns (18), (20) and (21) into the above equations yield

$$M(\theta)=\frac{EI}{R^2}\sum_{n=1}^{6} c_n m_n e^{\lambda_n\theta} \tag{30}$$

$$T(\theta)=\frac{EI}{R^2}\sum_{n=1}^{6} c_n t_n e^{\lambda_n\theta} \tag{31}$$

$$V(\theta)=\frac{EI}{R^3}\sum_{n=1}^{6} c_n v_n e^{\lambda_n\theta} \tag{32}$$

where

$$m_n=w_n-\lambda_n z_n,\quad t_n=\rho(z_n+\lambda_n w_n),\quad v_n=(\lambda_n-z_n)/s^2. \tag{33}$$

With reference again to Fig. 2, the boundary conditions are

$$\left.\begin{aligned}\Psi_a=\Psi(0),\ \Psi_b=\Psi(\alpha)\\ \Phi_a=\Phi(0),\ \Phi_b=\Phi(\alpha)\\ Y_a=Y(0),\ Y_b=Y(\alpha)\end{aligned}\right\}. \tag{34}$$

Similarly, the bending and twisting moments and shear forces at the two ends may be written as

$$\left.\begin{aligned} M_{ab} &= M(0), & M_{ba} &= -M(\alpha) \\ T_{ab} &= T(0), & T_{ba} &= -T(\alpha) \\ V_{ab} &= V(0), & V_{ba} &= V(\alpha) \end{aligned}\right\}. \tag{35}$$

The substitution of eqns (18), (20), (21) and (30)–(32) into eqns (34) and (35) will yield the following results in matrix forms:

$$\mathbf{D} = \mathbf{A}\mathbf{X} \tag{36}$$

$$\mathbf{F} = \left(\frac{EI}{R^3}\right)\mathbf{B}\mathbf{X} \tag{37}$$

where

$$\mathbf{D} = \begin{bmatrix} \Psi_a R \\ \Psi_b R \\ \Phi_a R \\ \Phi_b R \\ Y_a \\ Y_b \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} M_{ab}/R \\ M_{ba}/R \\ T_{ab}/R \\ T_{ba}/R \\ V_{ab} \\ V_{ba} \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} \tag{38}$$

$$\mathbf{A} = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ z_1 e^{\lambda_1\alpha} & z_2 e^{\lambda_2\alpha} & z_3 e^{\lambda_3\alpha} & z_4 e^{\lambda_4\alpha} & z_5 e^{\lambda_5\alpha} & z_6 e^{\lambda_6\alpha} \\ w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ w_1 e^{\lambda_1\alpha} & w_2 e^{\lambda_2\alpha} & w_3 e^{\lambda_3\alpha} & w_4 e^{\lambda_4\alpha} & w_5 e^{\lambda_5\alpha} & w_6 e^{\lambda_6\alpha} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ e^{\lambda_1\alpha} & e^{\lambda_2\alpha} & e^{\lambda_3\alpha} & e^{\lambda_4\alpha} & e^{\lambda_5\alpha} & e^{\lambda_6\alpha} \end{bmatrix} \tag{39}$$

$$\mathbf{B} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ -m_1 e^{\lambda_1\alpha} & -m_2 e^{\lambda_2\alpha} & -m_3 e^{\lambda_3\alpha} & -m_4 e^{\lambda_4\alpha} & -m_5 e^{\lambda_5\alpha} & -m_6 e^{\lambda_6\alpha} \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ -t_1 e^{\lambda_1\alpha} & -t_2 e^{\lambda_2\alpha} & -t_3 e^{\lambda_3\alpha} & -t_4 e^{\lambda_4\alpha} & -t_5 e^{\lambda_5\alpha} & -t_6 e^{\lambda_6\alpha} \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ v_1 e^{\lambda_1\alpha} & v_2 e^{\lambda_2\alpha} & v_3 e^{\lambda_3\alpha} & v_4 e^{\lambda_4\alpha} & v_5 e^{\lambda_5\alpha} & v_6 e^{\lambda_6\alpha} \end{bmatrix} \tag{40}$$

Eliminating **X** from eqns (36) and (37), the following equation is obtained:

$$\mathbf{F} = \mathbf{S}\mathbf{D} \tag{41}$$

where S, the dynamic stiffness matrix for a horizontally circular curved member, is given by

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix} = \left(\frac{EI}{R^3}\right)\mathbf{B}\mathbf{A}^{-1}. \tag{42}$$

4. NUMERICAL EXAMPLE

A three-span symmetrical circular curved beam A–B–C–D of constant section undergoing out-of-plane vibrations as shown in Fig. 4 is analyzed for natural frequencies. The beam is resting on rigid, non-twisting supports equally spaced at an angle α with the two extreme ends A and D hinged.

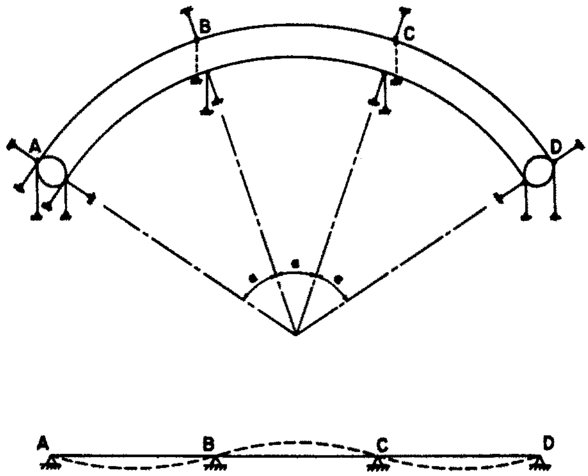

Fig. 4. Three-span circular curved beam.

Since no deflection or twist is allowed at the joints, each joint will have a rotation only. The equilibrium conditions at joints A, B, C and D give

$$M_{AB}=0,\quad M_{BA}+M_{BC}=0,\quad M_{CB}+M_{CD}=0,\quad M_{DC}=0. \tag{43}$$

From eqns (41) and (42) we have

$$\left.\begin{aligned}
M_{AB}/R &= s_{11}\Psi_A R + s_{12}\Psi_B R\\
M_{BA}/R &= s_{21}\Psi_A R + s_{22}\Psi_B R\\
M_{BC}/R &= s_{11}\Psi_B R + s_{12}\Psi_C R\\
M_{CB}/R &= s_{21}\Psi_B R + s_{22}\Psi_C R\\
M_{CD}/R &= s_{11}\Psi_C R + s_{12}\Psi_D R\\
M_{DC}/R &= s_{21}\Psi_C R + s_{22}\Psi_D R.
\end{aligned}\right\} \tag{44}$$

Upon substituting eqns (44) into eqns (43), a system of simultaneous equations in the following matrix form is obtained:

$$\begin{bmatrix} s_{11} & s_{12} & 0 & 0\\ s_{21} & s_{11}+s_{22} & s_{21} & 0\\ 0 & s_{21} & s_{11}+s_{22} & s_{12}\\ 0 & 0 & s_{21} & s_{22}\end{bmatrix}\begin{bmatrix}\Psi_A R\\ \Psi_B R\\ \Psi_C R\\ \Psi_D R\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}. \tag{45}$$

Setting the determinant of the stiffness matrix in eqn (45) to zero gives the frequency equation as

$$\begin{vmatrix} s_{11} & s_{12} & 0 & 0\\ s_{21} & s_{11}+s_{22} & s_{12} & 0\\ 0 & s_{21} & s_{11}+s_{22} & s_{12}\\ 0 & 0 & s_{21} & s_{22}\end{vmatrix}=0. \tag{46}$$

For a given curved beam with r and s known, the values of b can be computed from eqn (46). In order to show the effects of shear deformation, rotary inertia and the opening angle of the arc on the natural frequencies of the beam, the values of ρ and k are assumed to be 0.77 and 0.89, respectively, for a beam of solid circular section with Poisson's ratio

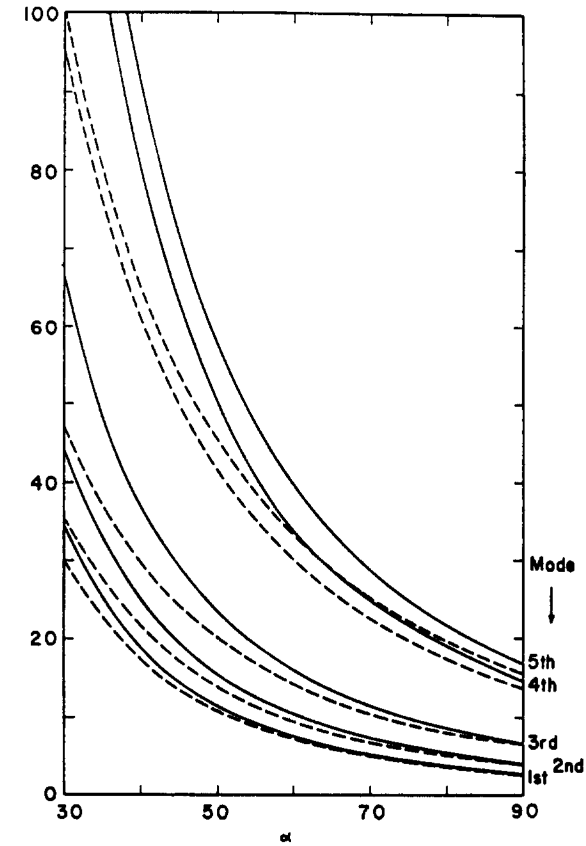

Fig. 5. Variation of b with α for a three-span curved beam. ———, $r = 0$; - - - -, $r = 0.05$.

$\mu = 0.3$[15]. Thus $s/r = \sqrt{E/(kG)} \approx 1.7$. The results of b vs α for $r = 0$ and 0.5 for the first five modes, with α varying from 30 to 90°, are plotted in Fig. 5.

5. CONCLUSIONS

The general matrix formulation for out-of-plane vibrations of circular curved members, including the effects of shear deformation and rotary inertia, has been presented for use in the determination of the natural frequencies of continuous curved beams. A three-span curved beam undergoing out-of-plane free vibrations is given to illustrate the application of the proposed method. From the results shown in Fig. 5, it is seen that the effects of shear deformation and rotary inertia are more important with increasing mode numbers and decreasing opening angles of the arc. For high modes the curves indicate that an increase in reduction of the ratio of natural frequencies between $r = 0$ and $r = 0.5$ as high as 40% is possible.

REFERENCES

1. J. P. Den Hartog, *Mechanical Vibrations*. McGraw-Hill, New York (1956).
2. E. Volterra and J. D. Morell, Lowest natural frequencies of elastic hinged arcs. *J. Acou. Soc. Am.* **33**, 1787–1790 (1961).
3. E. Volterra and J. D. Morell, Lowest natural frequency of elastic arc for vibrations outside the plane of initial curvature. *J. Appl. Mech.* **28**, 624–627 (1961).
4. T. M. Wang, Fundamental frequency of clamped elliptic arcs for vibrations outside the plane of initial curvature. *J. Sound Vib.* **42**, 515–519 (1975).
5. I. U. Ojalvo, Coupled twist-bending vibrations of incomplete elastic rings. *Int. J. Mech. Sci.* **4**, 53–72 (1962).
6. T. M. Wang, R. H. Nettleton and B. Keita, Natural frequencies for out-of-plane vibrations of continuous curved beams. *J. Sound Vib.* **68**, 427–436 (1980).
7. S. P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Phil. Mag.* **41**, 744–746 (1921).
8. L. L. Philipson, On the role of extension in the flexural vibrations of rings. *J. Appl. Mech.* **23**, 364–366 (1956).

9. B. S. Seidal and E. A. Erdelyi, On the vibrations of a thick ring in its own plane. *J. Engng Industry* **86**, 240–244 (1964).
10. S. S. Rao and V. Sundararajan, In-plane flexural vibrations of circular rings. *J. Appl. Mech.* **36**, 620–625 (1969).
11. S. S. Rao, Effects of transverse shear and rotatory inertia on the coupled twist-bending vibrations of circular rings. *J. Sound Vib.* **16**, 551–566 (1971).
12. T. M. Wang and M. Guilbert, Effects of rotary inertia and shear on natural frequencies of continuous circular curved beams. *Int. J. Solids Structures* **17**, 281–289 (1981).
13. E. Volterra and J. H. Gaines, *Advanced Strength of Material*. Prentice-Hall, New Jersey (1971).
14. S. P. Timoshenko, *Vibration Problems in Engineering*. D. Van Nostrand, New York (1955).
15. G. R. Cowper, The shear coefficient in Timoshenko's beam theory. *J. Appl. Mech.* **33**, 335–340 (1966).